Lecture 1 2012/09/17

Outline:

1. **Motivation for doing this course**
2. **Teaching program**
3. **About textbook**
4. **Banach Space: Basic Concepts; Fixed Point Theorems**

1. Motivation for doing this course:

1) To learn more about ODE, how its solutions behave, why it is important, what solution characteristics are, and how to deal with them, especially in geometric point of view.
2) To get a better understanding of modern ODE, e.g. local and global theory of nonlinear systems, geometric properties of linear systems, advanced stability theory.
3) To master some powerful tools used frequently in analysis of ODE, e.g. inequality techniques, geometric approaches, Lyapunov methods, etc.
4) To be introduced to some topics of ODE that would enable you to read or understand a lot of modern literatures for your future study at graduate level in pure, or applied Math.

2. Teaching program:

1) Part 1. Fundamental Theory (basically, local theory of nonlinear systems): including existence, uniqueness, continuation, continuous dependence, global existence, and many inequality techniques introduced.
2) Pare 2. Linear Systems: including general theory, stable and unstable theorem, Floquet theory and linearization.
3) Par 3. Stability Analysis: stable and unstable manifold theorem, center manifold, Lyapunov stability and LaSalle invariant principle.
4) Par 4. Poincare-Bendixson Theorem (global theory of nonlinear systems): limits, attractors, Poincare map, stability of periodic solutions, Poincare-Bendixson Theorem.

3. About textbook:

See Syllabus.

4. What to do and how to do:

See Syllabus.

Chapter 1 Fundamental Theory

1.1 Banach Space

1) Normed Linear Space

Let E be a linear space, equipped with a norm $\|\cdot\|: E \to R_{\geq 0}$, which satisfies three properties as follows:

(1) $\|x\| \geq 0$; $\|x\| = 0 \Leftrightarrow x = 0$ for any $x \in E$;

(2) $\|\lambda x\| = |\lambda| \|x\|$ for any $x \in E$, $\lambda \in R$;

(3) $\|x + y\| \leq \|x\| + \|y\|$ for any $x, y \in E$,

then, E is called a normed linear space.

Remark 1.1 $\|\cdot\|: E \to R_{\geq 0}$ is a natural extension of $|\cdot|: R \to R_{\geq 0}$. With $\|\cdot\|$, we are able to play "convergence" on E, not only on R^n like doing in Calculus.

2) Convergence, Cauchy Sequence

If $\forall\ \varepsilon > 0$, $\exists\ N \geq 1$ s.t. when $n \geq N \Rightarrow \|x_n - x\| < \varepsilon$, where $x_n, x \in E$, then, we say that x_n converges to x as $n \to \infty$, denoted by $\lim_{n\to\infty} x_n = x$.

If $\forall\ \varepsilon > 0$, $\exists\ N \geq 1$ s.t. $\|x_n - x_m\| < \varepsilon$ whenever $n, m \geq N$, where $x_n, x_m \in E$, then, we say that $\{x_n\}$ is a Cauchy sequence.

3) Banach Space

A normed linear space E is said to be complete if for every $\{x_n\} \subset E$ with $\lim_{n\to\infty} x_n = x \Rightarrow x \in E$.

A complete normed linear space E is said to be a Banach space.

Remark 1.2 Since the norm defined on E is not unique, notion of norm equivalence is significant. e.g. $E = R^n$ with $\|x\|_2 = (\sum_{i=1}^{n} x_j^2)^{1/2}$; $\|x\|_1 = \sum_{i=1}^{n} |x_j|$, and $\|x\|_\infty = \sup_j |x_j|$.

4) Norm Equivalence

Let $\|\cdot\|$ and $\|\cdot\|_*$ be two norms on E. $\|\cdot\|$ and $\|\cdot\|_*$ are equivalent if there exist $c>0$ and $\bar{c}>0$ s.t.

$$c\| x \| \le \| x \|_* \le \bar{c}\| x \|, \text{ for all } x \in E.$$

Remark 1.3 Since any $\|\cdot\|_q$, $1 \le q \le \infty$, are equivalent on R^n, R^n equipped with $\|x\|_q$ is a Banach space (easily checked).

5) Countexample for an Infinite Dimensional Banach Space

Proposition 1.1 Let $C([0,1]) = \{f \in R^n \text{ is continuous on } [0,1]\}$. $C([0,1])$ with $\| f \|_\infty = \sup_{t\in[0,1]} | f(t)|$ is a Banach space. However, $C([0,1])$ with either of the norms $\| f \|_1 = \int_0^1 | f(t)| dt$ or $\| f \|_2 = \left(\int_0^1 | f(t)|^2 dt\right)^{1/2}$ is not Banach.

Proof. Following the definition of a norm, we may verify easily that $\| f \|_1$, $\| f \|_2$ and $\| f \|_\infty$ are all norms. Omitted.

(1) To show that $C([0,1])$ with $\| f \|_\infty$ is Banach, it is suffice to show that for any Cauchy sequence $\{f_n\}$ in $C([0,1])$ with $\| f \|_\infty$, $\{f_n(t)\}$ is uniformly convergent to $f(t)$ on $[0,1]$ because $f_n(t)$ is continuous on $[0,1]$, so is $f(t)$ on $[0,1]$. Therefore, $f(t) \in C([0,1])$.

(2) It remains to show that $\{f_n(t)\}$ is uniformly convergent on $[0,1]$. Since $\{f_n\}$ is Cauchy sequence, by the definition of $\| f \|_\infty$, we have

$$| f_n(t) - f_m(t)| \le \sup_{t\in[0,1]} | f_n(t) - f_m(t)| = \| f_n - f_m \|_\infty < \varepsilon, \text{ for all } n, m \ge N, \quad (1.17)$$

it implies that $\{f_n(t)\}$ is Cauchy for any $t \in [0,1]$ in R, which is complet. Then, $\lim_{n\to\infty} f_n(t) = f(t)$ exists on $[0,1]$. In particular, (1.17) holds for $m \ge n \ge N$. Then taking $m \to \infty$ in (1.17), we have

$$| f_n(t) - f(t) | < \varepsilon \quad \text{for all} \quad n \geq N,$$

where N doesn't depend on $t \in [0,1]$. Therefore, $\{ f_n(t) \}$ is uniformly convergent to $f(t)$ on $[0,1]$.

(3) To show that $C([0,1])$ with $\| f \|_1$ and $\| f \|_2$ is not complet.

Define $f_n(t) = \begin{cases} 0, & [0, \frac{1}{2} - \frac{1}{n}] \\ 1, & [\frac{1}{2} + \frac{1}{n}, 1] \end{cases}$, $n = 1, 2, \cdots$, linearly interpolating in between.

Then, $f_n(t) \in C([0,1])$. and $\lim_{n \to \infty} f_n(t) = f(t) = \begin{cases} 0, & [0, \frac{1}{2}) \\ 1, & (\frac{1}{2}, 1] \end{cases}$.

According to the definitions of $\| f \|_1$ and $\| f \|_2$, it is easy to verify that for any $m \geq n \geq N$ we have

$$\| f_n - f_m \|_1 \leq \int_0^1 | f_n(t) - f_m(t) | \, dt = 2 \cdot \frac{1}{2} \cdot \frac{1}{2} \left(\frac{1}{n} - \frac{1}{m} \right) \leq \frac{1}{n}$$

and

$$\| f_n - f_m \|_2 = \{ \int_0^1 | f_n(t) - f_m(t) |^2 \, dt \}^{\frac{1}{2}} \leq \{ \int_0^1 | f_n(t) - f_m(t) | \, dt \}^{\frac{1}{2}} \leq \frac{1}{\sqrt{n}},$$

$$\Rightarrow \quad \| f_n - f \|_1 \leq \frac{1}{n} \quad \text{and} \quad \| f_n - f \|_2 \leq \frac{1}{\sqrt{n}}, \text{ if taking } m \to \infty;$$

$\{ f_n \}$ is Cauchy both with $\| f \|_1$ and $\| f \|_2$.

However, $f(t) = \begin{cases} 0, & [0, \frac{1}{2}) \\ 1, & (\frac{1}{2}, 1] \end{cases} \notin C([0,1])$, therefore, $C([0,1])$ with $\| f \|_1$ or $\| f \|_2$ is neither Banach. □

6) Some Important Results in Banach Space

Proposition 1.2 Let X be any set and consider the space

$$B(X) = \{ f : X \to R \, ; \, f \quad \text{is bounded} \}$$

with the norm $\| f \|_\infty = \sup_{x \in X} | f(x) |$. Then $B(X)$ is a Banach space.

Proof. Left for homework since its proof is similar to $C([0,1])$ with $\| f \|_\infty$.

Remark 1.4 In a Banach space, we can define basic topology as in R^n.

(1) A open ball of radius $r > 0$ and center $a \in E$ is $B_\varepsilon(a) = \{ x \in E; \| x - a \| < r \}$.

(2) A set $U \subset E$ is open if for any $x \in U$, $\exists\ \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset U$.

(3) A set $V \subset E$ is closed if the limit of any convergent sequence $\{x_n\}$ with $x_n \in V$ is in V.

(4) A set K is compact if any sequence $\{x_n\} \subset K$ has a subsequence which converges in K.

(5) Let E and F be two Banach spaces and $U \subset E$. A function $f : U \to F$ is said to be continuous at $x_0 \in U$ if $\forall\ \varepsilon > 0$, $\exists\ \ \delta > 0$ s.t.

$$\| f(x) - f(x_0) \| < \varepsilon ,$$

whenever $\| x - x_0 \| < \delta$, where $x \in U$.

(6) The map $\| \cdot \| : E \to R$ is continuous. (Left for homework)

Notice that it needs to show that $-\| x - x_0 \| \le \| x \| - \| x_0 \| \le \| x - x_0 \|$, which can be shown by the triangle inequality.

Remark 1.5 Some properties which are true in finite dimensional Banach spaces are not necessarily true in infinite dimensional Banach spaces! For example:

(1) The closed ball $\{ x \in E; \| x \| \le 1 \}$ is not necessarily compact!

(2) Two norms on an infinite dimensional Banach space are not always equivalent!

(3) The Bolzano-Weierstrass theorem which says each bounded sequence has a convergent subsequence is not necessarily true!

(4) "K is compact $\Leftrightarrow$ K is both closed and bounded" is not necessarily true in a Banach space!

Left for homework. Pay attention to the following example. Define

$$f_n(t) = \begin{cases} 0, & [0,1]/(\frac{1}{(n+1)}, \frac{1}{n}] \\ 1, & \frac{1}{(n+1)} < t \le \frac{1}{n} \end{cases}.$$

$f_n(t)$ is bounded on $[0,1]$, i.e. $f_n \in B([0,1])$; Since $\| f_m - f_n \|_\infty = 1$ for any n, m, $\{f_n\}$ is not Cauchy, which yields $\{f_n\}$ divergent. Hence, the bounded sequence $\{f_n\}$ has no convergent sub-sequence. Therefore, Bolzano- Weierstrass theorem fails. Also $\| f_n \|_\infty = 1$ shows (1) and (4).

7) Banach Fixed Point Theorem

Theorem 1.1 Let E be a Banach space with a norm $\|\cdot\|$, $D \subset E$ closed and $f : D \to E$ a map which satisfies

(1) $f(D) \subset D$;

(2) There exists $0 < \alpha < 1$ s.t. $\| f(x) - f(y) \| \le \alpha \| x - y \|$ for all $x, y \in D$.

Then f has a unique fixed point x in D, i.e. $f(x) = x$.

Proof. Step 1. Uniqueness.

If there exist $x, y \in D$ s.t. $f(x) = x$ and $f(y) = y$, then we have

$$\| x - y \| = \| f(x) - f(y) \| \le \alpha \| x - y \| \text{ with } 0 < \alpha < 1.$$

This is not possible unless $x = y$.

Step 2. Existence. (Constructive)

Taking any $x_0 \in D$, iteration yields the sequence

$$x_1 = f(x_0),\ x_2 = f(x_1),\ \cdots,\ x_{n+1} = f(x_n),\ \cdots.$$

Note that if there exists $k \ge 1$ s.t. $x_k = f(x_k)$, x_k is desired. If for any $k \ge 1$, $x_k \ne f(x_k)$.

The sequence $\{x_n\}$ is well defined on D since $x_n \in D$ by $f(D) \subset D$.

To show $\{x_n\}$ is Cauchy on D. Since

$$\| x_{n+1} - x_n \| = \| f(x_n) - f(x_{n-1}) \| \le \alpha \| x_n - x_{n-1} \|,$$

we have the following inequality by iterating

$$\| x_{n+1} - x_n \| \le \alpha^n \| x_1 - x_0 \|. \tag{1.24}$$

If $m > n$, this implies that

$$\begin{aligned} \| x_m - x_n \| &\le \| x_m - x_{m-1} \| + \| x_{m-1} - x_{m-2} \| + \cdots + \| x_{n+1} - x_n \| \\ &\le (\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n) \| x_1 - x_0 \| \\ &\le \frac{\alpha^n}{1-\alpha} \| x_1 - x_0 \| < \varepsilon \quad \text{if} \quad n > N = \left[\ln \frac{\varepsilon(1-\alpha)}{\| x_1 - x_0 \|} \Big/ \ln \alpha \right] + 1. \end{aligned}$$

Therefore, $\{x_n\}$ is Cauchy on D. $\Rightarrow$ $\lim_{n\to\infty} x_n = x \in D$ since D is closed in E, which is Banach. By the condition (2) we know that f is continuous. Then,

$$f(x) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x.$$

i.e. x is a fixed point of f. □

Remark 1.6 There are plenty of applications of Banach fixed point theorem. What we concern is its application in ODE. However, Banach fixed point theorem works only for the existence and uniqueness. For the existence without uniqueness, we need the other kind of fixed point theorem, which is given in the next lecture.